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TOTALLY GEODESIC FOLIATIONS ON 4-MANIFOLDS

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Abstract

We give a rather detailed description of the behavior of 2-dimensional totally geodesic foliations on compact Riemannian 4-manifolds. In particular, we obtain a complete characterization in the simply connected case.

0. Introduction

A foliation $\mathscr{F}$ is "geodesible" if there exists a Riemannian metric that makes $\mathscr{F}$ totally geodesic. The problem of characterizing geodesible foliations has been essentially solved in the 1-dimensional case [16]. There exists in fact numerous geodesible flows; for instance, every compact 3-manifold admits contact flows and such flows are geodesible [11]. Equally, the problem has been solved in the codimension one case [10]. Here the situation is so rigid that one can give a complete classification. The basic point is that, in codimension 1, the distribution $\mathscr{F}^{\perp}$, orthogonal to $\mathscr{F}$, is evidently integrable. In arbitrary codimension, there is a global description of the qualitative behavior of geodesible foliations that tries to take into account the nonintegrability of $\mathscr{F}^{\perp}$ [2], [4], [6]. However, this method cannot give rise to a complete classification. The first case where this analysis provides effective tools is that of 2-dimensional foliations on 4-manifolds. This is the object of study of this paper.

We assume that the foliation $\mathscr{F}$ and the manifold M are oriented and C^{∞} (for some general comments concerning the C^0 case, see [20]).

The following theorem has the advantage of splitting the problem into three subproblems. Note that, for the present, we treat 2-dimensional foliations of arbitrary codimension.

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Theorem A. *Let $\mathcal{F}$ be a 2-dimensional geodesible foliation on a compact manifold M. Then there exists a Riemannian metric g on M for which $\mathcal{F}$ is totally geodesic and such that the curvature of the leaves is the same constant K, equal to* $+1$, 0, *or* -1.

So there are three types of geodesible 2-dimensional foliations that we will call respectively *elliptic* ($K = +1$), *parabolic* ($K = 0$) or *hyperbolic* ($K = -1$). The following three theorems treat these three cases.

Theorem B (*elliptic case*). *Let $\mathcal{F}$ be a 2-dimensional foliation on a compact manifold M. Then $\mathcal{F}$ is elliptic geodesible if and only if the leaves of $\mathcal{F}$ are the fibers of a fibration of M by spheres* $\mathbf{S}^2$.

Theorem C (*parabolic case*). *Let $\mathcal{F}$ be a parabolic geodesible 2-dimensional foliation on a compact 4-manifold M. Then there exists an Abelian covering $\hat{M}$ of M such that the lift $\hat{\mathcal{F}}$ of $\mathcal{F}$ to $\hat{M}$ can be defined by a locally free action of* $\mathbf{R}^2$.

Further results concerning the parabolic case will be given in §4. In particular, we describe the dynamics of $\mathcal{F}$ when the leaves are dense (see Theorem 4.1).

Theorem D (*hyperbolic case*). *Let $\mathcal{F}$ be a totally geodesic 2-dimensional foliation on a compact Riemannian 4-manifold M. If the leaves have constant negative curvature, the orthogonal distribution $\mathcal{F}^{\perp}$ is necessarily integrable. There are two possibilities:*

(1) *either $\mathcal{F}$ is defined by a suspension of a representation of the fundamental group of some surface of genus greater than one in the group of diffeomorphisms of the sphere* $\mathbf{S}^2$, *or*

(2) *the universal covering space $\tilde{M}$ of M is diffeomorphic to* $\mathbf{R}^4$, *in such a way that the leaves of $\mathcal{F}$ are covered by $\mathbf{R}^2 \times \{*\}$ and those of $\mathcal{F}^{\perp}$ by* $\{*\} \times \mathbf{R}^2$.

These results enable us to deduce the following theorems:

Theorem E. *Let M be a compact simply connected 4-manifold. If there exists a geodesible foliation $\mathcal{F}$ on M, then M is one of the two fibrations by spheres $\mathbf{S}^2$ over $\mathbf{S}^2$, and the leaves of $\mathcal{F}$ are the fibers of this fibration.*

Theorem F. *Let $\mathcal{F}$ be a 2-dimensional geodesible foliation on a compact 4-manifold M with negative Euler characteristic. Then M is a fibration by spheres $\mathbf{S}^2$ over a compact surface. Moreover, one can choose this fibration in such a way that its fibers are either everywhere tangent or everywhere transverse to $\mathcal{F}$.*

This paper is organized in the following manner. In the first section we recall the basic facts concerning geodesible foliations and we prove Theorems A and B. §2 provides some examples. In §3, we introduce the notion of "transverse curvature" which leads to the proof of Theorems C and D in §4. Finally, §5 is devoted to Theorems E and F.

We would like to acknowledge the friendly open assistance of Pierre Molino.

1. Preliminary results: Proofs of Theorems A and B

There are several ways to characterize totally geodesic foliations (see [13] for details). Roughly speaking, a foliation $\mathscr{F}$ is totally geodesic if and only if the "holonomy" of the orthogonal distribution $\mathscr{F}^{\perp}$ consists of local isometries. More precisely, let $\mathscr{F}$ be an arbitrary foliation on a compact Riemannian manifold (M, g), and let γ: $[0,1] \to M$ be an arc orthogonal to $\mathscr{F}$. Then, if V_0 is a sufficiently small neighborhood of $\gamma(0)$ in the leaf containing $\gamma(0)$, one can construct a unique mapping

$$(t, m) \in [0,1] \times V_0 \mapsto F_t(m) \in M$$

such that

(i) F_0: $V_0 \to M$ is the natural injection,

(ii) $F_t(\gamma(0)) = \gamma(t)$,

(iii) F_t: $V_0 \to M$ is a diffeomorphism onto a neighborhood of $\gamma(t)$ in the leaf containing $\gamma(t)$.

(iv) the curves $t \mapsto F_t(m)$ are orthogonal to $\mathscr{F}$.

The foliation $\mathscr{F}$ is totally geodesic for g if and only if all these local diffeomorphisms F_t, corresponding to the different choices of γ, are isometries (see [2]).

As a simple application of this criteria, we have

Proposition 1.1 (*see also* [19]). *A locally trivial fibration with compact fiber is geodesible if and only if its structure group can be reduced to a group of isometries.*

We can now prove Theorem B.

Proof of Theorem B. If $\mathscr{F}$ is elliptic geodesible, then $\mathscr{F}$, being supposed orientable, is obviously a foliation by spheres and therefore a fibration by the Reeb stability theorem. Conversely, in view of Proposition 1.1, Theorem B follows from Smale's theorem (see [8]) by which $O(3, \mathbf{R}) \simeq \mathrm{Isom}(\mathbf{S}^2, can)$ is a deformation retract of $\mathrm{Diff}(\mathbf{S}^2)$. q.e.d.

Before continuing, let us recall some notations and results from [2], [4], [5]. If m is a point of M, the *sheet* $S(m)$ through m is the set of points of M that can be reached from m by piecewise smooth paths orthogonal to $\mathscr{F}$. A totally geodesic foliation $\mathscr{F}$ of dimension p is said to be *tangentially parallelizable* if there exists p orthonormal vector fields, tangent to the foliation $X_1, \cdots, X_p$, whose flows preserve $\mathscr{F}^{\perp}$. If, further, the $X_1, \cdots, X_p$ generate a p-dimensional Lie algebra $\mathfrak{G}$, the foliation is called a *tangentially* $\mathfrak{G}$*-Lie foliation*.

In this case, $\mathscr{F}$ can be defined by the action of the corresponding simply connected Lie group. One has

Theorem 1.2 ([2], [4], [6]). *Let $\mathscr{F}$ be a p-dimensional totally geodesic foliation on a compact connected Riemannian manifold (M, g). The pull-back $\hat{\mathscr{F}}$ of $\mathscr{F}$ to the principal* SO(*p*)*-bundle E of orthonormal oriented tangent p-frames is canonically tangentially parallelizable* (*with respect to the canonical lift of g*). *The sheets of $\hat{\mathscr{F}}$ define a Riemannian foliation whose leaves project in M onto the sheets of $\mathscr{F}$. Moreover, the sheets of $\mathscr{F}$, and their closures, define respectively "singular foliations."*

Theorem 1.3 ([5]). *If $\mathscr{F}$ is a 2-dimensional totally geodesic foliation on a compact connected Riemannian manifold (M, g), then there are four possibilities*:

(i) *a sheet is dense, in which case all the leaves have the same constant curvature*,

(ii) *a finite nonzero number of sheets are compact and of codimension* 2,

(iii) *all the sheets are compact and of codimension* 2,

(iv) *the closures of the sheets define a codimension* 1 *foliation, in which case, up to a* 2*-fold cover, there is a Riemannian fibration of M onto* $\mathbf{S}^1$ *transverse to* $\mathscr{F}$.

We can now begin the proof of Theorem A. Obviously, we need only consider the cases (ii), (iii), and (iv) of the previous theorem, which will be treated by the following three propositions.

Proposition 1.4. *Let $\mathscr{F}$ be a* 2*-dimensional totally geodesic foliation of type* (ii). *Then all the leaves of $\mathscr{F}$ are in fact spheres and, in this case, Theorem* A *follows from Theorem* B.

Proof. Let L be a leaf of $\mathscr{F}$. The traces of the closures of the sheets on L define a singular foliation $\mathscr{F}_L$. The singularities of $\mathscr{F}_L$ correspond to the codimension 2 compact sheets. One verifies easily that these singularities are all centers; that is, around each singularity, $\mathscr{F}_L$ looks like a family of concentric circles. Away from the singularities, the leaves of $\mathscr{F}_L$ are circles. The only surfaces admitting such a singular foliation, with at least one singular point, are the sphere $\mathbf{S}^2$ (two singular points) and the plane $\mathbf{R}^2$ (one singular point). Then, as each leaf meets every sheet, $\mathscr{F}_L$ possesses at least one singular point, and therefore at most two. In order to establish the proposition, it remains to show that L cannot be the plane. But if L were noncompact, it would meet each sheet infinitely often, hence producing infinitely many singular points. q.e.d.

Proposition 1.5. *Theorem* A *holds for* 2*-dimensional totally geodesic foliations of type* (iii).

Proof. We use here a method employed in [7]. In our case, $\mathscr{F}^{\perp}$ is clearly integrable. The universal covering space $\tilde{M}$ of M is diffeomorphic to the product $\tilde{L}_1 \times \tilde{L}_2$, where $\tilde{L}_1$ (resp. $\tilde{L}_2$) is the universal cover of a leaf (resp. sheet) of $\mathscr{F}$ (see [1]). The fundamental group Γ of M acts by isometries on the $(\tilde{L}_1, g_1)$ factor, where g_1 is the natural metric on $\tilde{L}_1$. We change g_1 conformally to obtain a new metric g_1' of constant curvature K, equal to -1, 0, or $+1$. When K is -1, $(\tilde{L}_1, g_1')$ is identified with the Poincaré disc $\mathbf{D}^2$ and Γ acts by conformal diffeomorphisms of $\mathbf{D}^2$, and hence by isometries. This enables us to define, for $K = -1$, the required metric on M. For $K = 0$, $(\tilde{L}_1, g_1')$ is identified with the Euclidean plane $\mathbf{E}^2$, and Γ acts by conformal diffeomorphisms, that is, by similarities. Every similarity which is not an isometry has necessarily an attractive or repulsive fixed point. As this is not possible for a g_1-isometry, we conclude that Γ acts by isometries on $(\tilde{L}_1, g_1')$, and we obtain once again the required metric on M. The last case, $K = +1$, was already resolved by Theorem B. q.e.d

Proposition 1.6. *Theorem* A *holds for 2-dimensional totally geodesic foliations of type* (iv).

Proof. First assume that the closures of the sheets define a Riemannian fibration $p\colon M \to \mathbf{S}^1$. Let X be the vector field tangent to $\mathscr{F}$ that projects onto $\partial/\partial\theta$ and which is orthogonal to the fibers of p. Let Y be the vector field tangent to $\mathscr{F}$ such that (X, Y) is an orthonormal frame, positive for the orientation of $\mathscr{F}$. Clearly $[X, Y] = fY$, where f is a function constant along the sheets. Therefore, there exists a function $h\colon \mathbf{S}^1 \to \mathbf{R}$ such that $f = h \circ p$.

It is elementary to check that there is a function $u\colon \mathbf{S}^1 \to \mathbf{R}$ such that

$$\frac{\partial}{\partial\theta}u + h = \text{Constant}.$$

Then consider the vector field $\overline{Y} = \exp(u \circ p)Y$. One has

$$(*) \qquad [X, \overline{Y}] = (f + X(u \circ p))\overline{Y} = k\overline{Y},$$

where k is a constant. We now redefine the metric on $\mathscr{F}$ such that $(X, \overline{Y})$ forms an orthonormal frame. It is clear $\mathscr{F}$ is still totally geodesic and the fact that k is constant implies that $\mathscr{F}$ has constant curvature (parabolic if $k = 0$, and hyperbolic otherwise).

It remains to consider the case where the codimension 1 foliation defined by the closures of the sheets is not orientable. In this case, there is a double cover $\hat{M}$, equipped with the standard involution τ, for which the above construction holds. One verifies that $\tau_* X = -X$, $\tau_* Y = -Y$, and $\tau_* \overline{Y} = -\overline{Y}$ and so the change of metric in $\hat{M}$ is τ-invariant. This completes the proof of the proposition. Note that applying τ to the relation $(*)$, one obtains $k = 0$ and so the latter case is necessarily parabolic. q.e.d.

This completes the proof of Theorem A.

Proposition 1.7. *The properties of being elliptic, parabolic or hyperbolic define a partition on the set of geodesible foliations on compact manifolds.*

Proof. Since $\mathbf{S}^2$ has no metric of constant curvature equal to 0 or -1, no elliptic geodesible foliation can be parabolic or hyperbolic. Now, the leaves of a parabolic foliation, being flat complete surfaces, are diffeomorphic to $\mathbf{R}^2$, $\mathbf{R} \times \mathbf{S}^1$, or $\mathbf{T}^2$ and have polynomial growth (cf. [15]). There is no metric on T^2 with curvature -1 and the complete metrics on $\mathbf{R}^2$ and $\mathbf{R} \times \mathbf{S}^1$ with curvature -1 have exponential growth. So a parabolic geodesible foliation cannot be hyperbolic. q.e.d.

2. Examples

Let us first of all recall that the problem of the existence of 2-dimensional foliations on 4-manifolds is completely understood. In fact, according to [17], every 2-dimensional plane field is homotopic to a foliation and the existence of plane fields is expressed by an algebraic condition involving the intersection form, the signature, and the Euler characteristic [12].

Example 2.1: Suspension of a geodesible flow. Let Y be a vector field defining a flow by geodesics on a Riemannian 3-manifold M^3. Let ϕ be a diffeomorphism of M^3 such that $\phi_* Y = \lambda Y$ for some positive constant λ. Consider the 4-manifold M^4 defined by:

$$M^4 = M^3 \times [0,1]/(m,0) \sim (\phi(m),1).$$

There exists a natural 2-dimensional foliation on M^4 which is geodesible if ϕ preserves the plane field orthogonal to Y.

Recalling that any 3-manifold admits a geodesible flow and choosing $\phi = \mathrm{id}$, we see that any product $M^3 \times \mathbf{S}^1$ admits a geodesible 2-dimensional foliation. Another typical example is obtained when M^3 is a flat torus T^3. In this example, ϕ can be chosen as being the linear automorphism of $\mathbf{T}^3$ corresponding to a matrix of $\mathrm{SL}(3, \mathbf{Z})$ and Y is the linear vector field corresponding to an eigenvalue of this matrix.

Example 2.2: Linear foliations. Let $G_{3,2}$ denote the Grassmannian of 2-planes in $\mathbf{R}^3$, and let $\psi\colon \mathbf{S}^1 \to G_{3,2}$ be any smooth map. Consider the flat torus $\mathbf{T}^4$ equipped with the following 2-dimensional foliation $\mathscr{F}$: All the leaves of $\mathscr{F}$ are contained in $\mathbf{T}^3 \times \{*\}$ and the restriction of $\mathscr{F}$ to $\mathbf{T}^3 \times \{*\}$ is the usual linear foliation of $\mathbf{T}^3$ by planes parallel to $\psi(*)$. This foliation is obviously parabolic geodesible.

Example 2.3: Suspensions. Let Σ_1 and Σ_2 be two compact orientable surfaces, and let h be a representation of $\pi_1(\Sigma_1)$ in $\mathrm{Diff}^+(\Sigma_2)$. Then one constructs, in a well-known manner, a Σ_2-bundle over Σ_1 with a "suspended" foliation $\mathscr{F}$ transverse to the fibers. Noting that we can choose a metric such that this fibration is Riemannian, it is clear that $\mathscr{F}$ is geodesible. The foliation $\mathscr{F}$ is elliptic, parabolic, or hyperbolic, according to whether the genus of Σ_1 is zero, one, or greater than one.

Example 2.4: Local products of totally geodesic foliations. According to [3], there exists an irreducible lattice Γ in $\mathrm{PSL}(2, \mathbf{R}) \times \mathrm{PSL}(2, \mathbf{R})$ with compact quotient. The irreducibility of Γ means that there is no subgroup of finite index in Γ which is a product of two lattices of $\mathrm{PSL}(2, \mathbf{R})$. Moreover, by [3], we can assume that Γ is torsion free. Regarding $\mathrm{PSL}(2, \mathbf{R})$ as being the isometry group of the Poincaré disc $\mathbf{D}^2$, we obtain an action of Γ on the Riemannian product $\mathbf{D}^2 \times \mathbf{D}^2$ which is free because Γ is torsion free. The quotient of $\mathbf{D}^2 \times \mathbf{D}^2$ by this action is a 4-manifold M equipped with two orthogonal totally geodesic hyperbolic foliations $\mathscr{F}_1$ and $\mathscr{F}_2$. The sheets of $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) are the leaves of $\mathscr{F}_2$ (resp. $\mathscr{F}_1$).

We claim that the leaves of $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) are dense in M. Let pr_1 and pr_2 denote the two projections of $\mathrm{PSL}(2, \mathbf{R}) \times \mathrm{PSL}(2, \mathbf{R})$ on $\mathrm{PSL}(2, \mathbf{R})$. We have to show that $\mathrm{pr}_1(\Gamma)$ and $\mathrm{pr}_2(\Gamma)$ are dense in $\mathrm{PSL}(2, \mathbf{R})$. If one looks closely at the construction of Γ, one sees that pr_1 and pr_2 are injective when restricted to Γ. Note that a subgroup of $\mathrm{PSL}(2, \mathbf{R})$ is either dense, solvable, or discrete and that a lattice in a semi-simple group is never solvable. So, it suffices to show that $\mathrm{pr}_1(\Gamma)$ (resp. $\mathrm{pr}_2(\Gamma)$) is not a discrete subgroup of $\mathrm{PSL}(2, \mathbf{R})$. But, the cohomological dimension of Γ is obviously four, so Γ cannot be isomorphic to a discrete torsion free subgroup of $\mathrm{PSL}(2, \mathbf{R})$.

3. Transverse curvature

Let G be a simply connected Lie group and $\mathfrak{G}$ its Lie algebra of left invariant vector fields. Suppose $\mathscr{F}$ is a tangentially $\mathfrak{G}$-Lie foliation on a compact manifold M. One can regard $\mathscr{F}$ as a principal G-bundle, with the exception that there is no base: there is a locally free right action of G on M and the orthogonal distribution $\mathscr{F}^{\perp}$ is the analogue of a connection. Following P. Molino [14], we define the transverse curvature Ω as the element of $\Lambda^2(\mathscr{F}^{\perp}, \mathfrak{G})$ which measures the nonintegrability of $\mathscr{F}^{\perp}$. More precisely, if X and Y are two vector fields tangent to $\mathscr{F}^{\perp}$ and m is a point in M, then $\Omega_m(X, Y)$ is the orthogonal projection of $[Y, X]_m$ into $T_m\mathscr{F}$, seen as being isomorphic to $\mathfrak{G}$. The following lemma is standard.

Lemma 3.1. *The transverse curvature is pseudo-tensorial; that is, for all $g \in G$, one has $g^*\Omega = \mathrm{Ad}(g^{-1})\Omega$, where $g^*\Omega$ is the pull-back of Ω by the action of G, and $\mathrm{Ad}(g)$ is the standard adjoint representation.*

From now on, we assume that $\mathscr{F}$ is of codimension 2 and that $\mathfrak{G}$ is unimodular. Let w be a field of 2-vectors tangent to $\mathscr{F}^{\perp}$ that defines an orientation of $\mathscr{F}^{\perp}$. Then we define two maps:

(1) For all $g \in G$, J_g: $M \to \mathbf{R}_+^*$ is the "Jacobian" defined by $J_g(m)w_{m \cdot g} = (g_* w)_{m \cdot g}$, where $g_* w$ is the push-forward of w by g.

(2) Φ: $M \to \mathfrak{G}$, defined by $\Phi(m) = \Omega_m(w_m)$.

Lemma 3.2. *For all $g \in G$ and $m \in M$, we have:*

$$\Phi(m \cdot g) = J_g(m)^{-1}\mathrm{Ad}(g^{-1})(\Phi(m)).$$

Proof. Indeed,

$$\begin{aligned} J_g(m)\Phi(m \cdot g) &= J_g(m)\Omega_{m \cdot g}(w_{m \cdot g}) = \Omega_{m \cdot g}((g_* w)_{m \cdot g}) \\ &= \mathrm{Ad}(g^{-1})(\Phi(m)). \quad \text{q.e.d.} \end{aligned}$$

Denote by p the dimension of $\mathfrak{G}$, and choose a G-invariant field v of p-vectors tangent to $\mathscr{F}$ and positive for the orientation of $\mathscr{F}$. Such a field exists because $\mathfrak{G}$ is unimodular. Now let vol be the volume form on M such that $\mathrm{vol}(w \wedge v) = 1$. As M is compact, and since its total volume is preserved by the action of G, we have:

Lemma 3.3. *If $g \in G$, then $\int_M J_g \,\mathrm{vol} = \int_M \mathrm{vol}$.*

We now introduce:

Definition 3.4. If $\mathfrak{G}$ is an arbitrary Lie algebra, the *approximative center* $\mathscr{C}$ of $\mathfrak{G}$ is the complement of the set of points x of $\mathfrak{G}$ for which there exists a neighborhood U of x in $\mathfrak{G}$ and an element y of $\mathfrak{G}$ such that the maps

$$\mathrm{Ad}(\exp(ty))|_U\colon U \to \mathfrak{G}$$

converge uniformly towards infinity as t goes to positive infinity.

Observe that $\mathscr{C}$ contains the center of $\mathfrak{G}$. Rougly speaking, $\mathscr{C}$ consists of the elements that commute, up to "bounded terms," with all one-parameter subgroups.

The key result for the next section is the following.

Proposition 3.5. *Let $\mathscr{F}$ be a tangentially $\mathfrak{G}$-Lie foliation of codimension 2 on a compact manifold M and suppose that $\mathfrak{G}$ is unimodular. Then the image of the transverse curvature Ω lies in the approximative center $\mathscr{C}$ of $\mathfrak{G}$.*

Proof. First note that, as $\mathscr{C}$ is a closed cone, it suffices to show that the image of Φ lies in $\mathscr{C}$. Suppose that, for some point m_0 of M, we have $\Phi(m_0) \notin \mathscr{C}$. Then by definition, there exists an open neighborhood U of $\Phi(m_0)$ in $\mathfrak{G}$ and an element y in $\mathfrak{G}$ such that the maps

$$\mathrm{Ad}(\exp(ty))|_U\colon U \to \mathfrak{G}$$

tend to infinity as t goes to $+\infty$.

Let V be the inverse image of U by Φ. Then, by Lemma 3.2, we have

$$\Phi(m \cdot \exp(ty)) = \left(J_{\exp(ty)}(m)\right)^{-1}\mathrm{Ad}(\exp(-ty))(\Phi(m)).$$

As M is compact, the left side of this expression is bounded (as t tends to $-\infty$). By hypothesis, $\mathrm{Ad}(\exp(-ty))\Phi(m))$ goes uniformly to infinity on V (as t tends to $-\infty$) and so we conclude that $J_{\exp(ty)}(m)$ goes also uniformly to infinity on V. In particular,

$$\int_V J_{\exp(ty)}\mathrm{vol} \to +\infty \quad \text{as } t \to -\infty.$$

But, this is impossible in view of Lemma 3.3. q.e.d.

4. Proof of Theorems C and D

We begin by some general comments. Let $\mathscr{F}$ be a totally geodesic 2-dimensional foliation on a compact connected Riemannian manifold (M, g), and let $\hat{\mathscr{F}}$ be the pull-back of $\mathscr{F}$ to the bundle E of orthonormal 2-frames tangent to $\mathscr{F}$ and positive. By Theorem 1.2, $\hat{\mathscr{F}}$ is tangentially parallelizable. Indeed, if the leaves have constant curvature K, the commutator coefficients of the vector fields defining the canonical tangential parallelism are constants. Thus, $\hat{\mathscr{F}}$ is tangentially Lie. It is obvious that, when $K = -1$, the corresponding Lie group is the universal cover $\widetilde{\mathrm{SL}}(2, \mathbf{R})$ of $\mathrm{SL}(2, \mathbf{R})$ and, when $K = 0$, the corresponding Lie group is the universal cover of the group of isometries of the Euclidean plane $\mathbf{E}^2$. Note as well that $\mathscr{F}^{\perp}$ is integrable if and only if $\hat{\mathscr{F}}^{\perp}$ is integrable.

Proof of Theorem D. Suppose that $\mathscr{F}$ is hyperbolic. By Proposition 3.5, in order to show that $\mathscr{F}^{\perp}$ is integrable, it suffices to prove that the approximative center $\mathscr{C}$ of $\mathrm{sl}(2, \mathbf{R})$ is $\{0\}$.

The orbits of the adjoint representation of $\widetilde{\mathrm{SL}(2, \mathbf{R})}$ look like Figure 1. As the only compact orbit is $\{0\}$, this suggests the result. Explicitly, let $\{X, Y, Z\}$ be a basis of the algebra $\mathrm{sl}(2, \mathbf{R})$ such that:

$$[X, Y] = Y, \quad [X, Z] = -Z, \quad [Y, Z] = X.$$

For this basis, one has

$$\mathrm{Ad}(\exp(tX)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(t) & 0 \\ 0 & 0 & \exp(-t) \end{pmatrix}.$$

Consequently, $\mathscr{C}$ is contained in $\mathbf{R}X \oplus \mathbf{R}Z$. By considering $\mathrm{Ad}(\exp(-tX))$ one has $\mathscr{C} \subset \mathbf{R}X \oplus \mathbf{R}Y$, and so $\mathscr{C} \subset \mathbf{R}X$. Then, as $\mathscr{C}$ is invariant by $\mathrm{Ad}(g)$ for all $g \in \widetilde{\mathrm{SL}(2, \mathbf{R})}$, and as $\mathbf{R}X$ is not invariant by $\mathrm{Ad}(\exp(tY))$, we conclude that $\mathscr{C}$ equals $\{0\}$. This proves the first part of Theorem D.

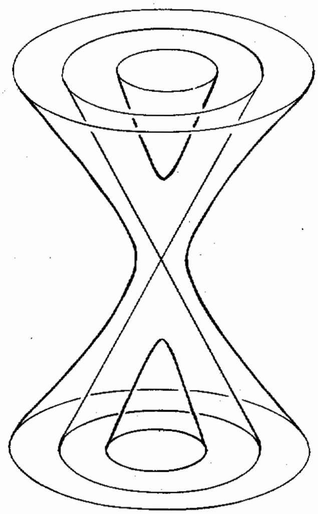

FIGURE 1

In [1], it is shown that if $\mathscr{F}^{\perp}$ is integrable, then the universal covering space $\tilde{M}$ of M is a product $\tilde{L}_1 \times \tilde{L}_2$, where $\tilde{L}_1$ (resp. $\tilde{L}_2$) is the universal cover of the leaf (resp. sheet) of $\mathscr{F}$. In our case, $\tilde{L}_1$ is diffeomorphic to $\mathbf{R}^2$. As well, $\mathscr{F}^{\perp}$ is 2-dimensional, and so $\tilde{L}$, is diffeomorphic to either $\mathbf{S}^2$ or $\mathbf{R}^2$. In the first case, $\mathscr{F}^{\perp}$ is a fibration by spheres (case (1) of Theorem D), and in the second case, $\tilde{M}$ is diffeomorphic to $\mathbf{R}^4$ (case (2) of Theorem D).

Proof of Theorem C. Now suppose that $\mathscr{F}$ is parabolic. Let us first compute the approximative center of the Lie algebra of the group of isometries of $\mathbf{E}^2$. This Lie algebra is generated by the basis $\{X, Y, Z\}$, where

$$[X,Y]=0, \quad [X,Z]=Y, \quad [Y,Z]=-X.$$

Here X and Y correspond to translations and Z to a rotation. The picture of the orbits of the adjoint representation is shown in Figure 2. The only bounded orbits are those contained in $\mathbf{R}X \oplus \mathbf{R}Y$. The reader can readily verify that indeed $\mathscr{C} = \mathbf{R}X \oplus \mathbf{R}Y$.

We now return to E, the SO(2, $\mathbf{R}$)-bundle of positive orthonormal tangent 2-frames of $\mathscr{F}$. Recall that $\{X, Y, Z\}$ defines a tangential parallelism of $\hat{\mathscr{F}}$. Clearly, Z is tangent to the fibers of the bundle E. By Proposition 3.5, the transverse curvature of $\hat{\mathscr{F}}$ has values in $\mathbf{R}X \oplus \mathbf{R}Y$. In other words, the distribution $\hat{\mathscr{F}}^{\perp} \oplus \mathbf{R}X \oplus \mathbf{R}Y$ is integrable. Any leaf $\hat{M}$ of the associated 4-dimensional foliation is transverse to Z and so covers M. The group of deck transformations is contained in $\mathbf{R}Z$ and is hence Abelian. It is now clear that

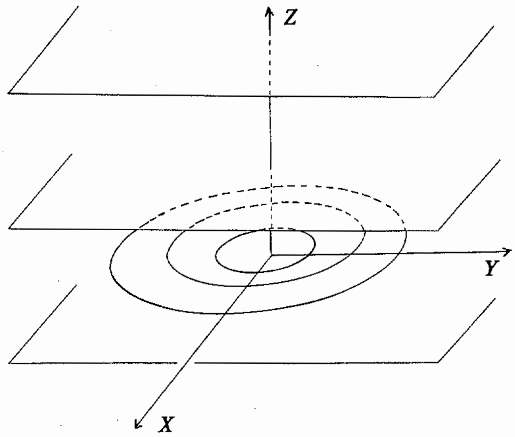

FIGURE 2

the lift of $\mathscr{F}$ to $\hat{M}$ is defined by the locally free action of $\mathbf{R}^2$ generated by X and Y. q.e.d.

As promised in the introduction, we now give further information concerning parabolic geodesible foliations.

Theorem 4.1. *Let $\mathscr{F}$ be a 2-dimensional totally geodesic parabolic foliation on a compact Riemannian manifold M. If all the leaves are dense, then there are two possibilities*:

(1) *either $\mathscr{F}^{\perp}$ is integrable, in which case the lifted foliation in the universal covering space of M is a product, or*

(2) *$\mathscr{F}$ is defined by the suspension of a geodesible flow, as in Example* 2.1. q.e.d.

Proof. Because of Lemma 3.2, the map Φ is everywhere or nowhere zero, since the leaves of $\hat{\mathscr{F}}$ are also dense. Case (1) corresponds to Φ being identically zero. Suppose then that Φ is nowhere zero. Let $\hat{\xi}$ be the unit vector field tangent to $\hat{\mathscr{F}}$ such that the image of Φ lies in $\mathbf{R}\hat{\xi}$. From the density of the leaves, and Lemma 3.2, we conclude that the flow of $\hat{\xi}$ preserves $\hat{\mathscr{F}}^{\perp}$. In particular, $\hat{\mathscr{F}}^{\perp} \oplus R\hat{\xi}$ defines a codimension 2 foliation. Evidently the leaves of this foliation are the sheets of $\hat{\mathscr{F}}$. One verifies that $\hat{\xi}$ is invariant by the action of $\mathrm{SO}(2, \mathbf{R})$ on E and defines a vector field ξ on M. Obviously, the sheets of $\mathscr{F}$ are the leaves of a foliation $\mathscr{N}$, tangent to $\mathscr{F}^{\perp} \oplus R\xi$, and so $\mathscr{N}$ is defined by a closed 1-form ω. According to Tischler's theorem [18], there exists a fibration of M onto $\mathbf{S}^1$ whose fibers approximate $\mathscr{N}$. So this fibration is transverse to $\mathscr{F}$. It is easy to verify that the trace of $\mathscr{F}$ on the fibers define a geodesible flow and $\mathscr{F}$ can be constructed by suspension of this flow, as in Example 2.1. q.e.d.

5. Applications

This section is devoted to the proof of Theorems E and F. Let us begin by the following preliminary result.

Lemma 5.1. *Let M be a compact 4-manifold. If there exists a parabolic geodesible foliation $\mathcal{F}$ on M, then the Euler characteristic $\chi(M)$ of M is zero.*

Proof. The Pfaffian of the curvature of a metric making $\mathcal{F}$ parabolic is identically zero. So, the Gauss-Bonnet-Chern theorem gives us the lemma. q.e.d.

Note that the Euler characteristic of a simply connected 4-manifold M is positive since it is equal to $2 + \dim H^2(M)$. Therefore, Theorem E will be a corollary of the following slightly more general theorem.

Theorem 5.2. *Let M be a compact 4-manifold with positive Euler characteristic and Abelian fundamental group. If there exists a geodesible foliation $\mathcal{F}$ on M, then M is one of the two fibrations by spheres $\mathbf{S}^2$ over $\mathbf{S}^2$ and the leaves of $\mathcal{F}$ are the fibers of this fibration.*

Proof. As $\chi(M)$ is nonzero, there is no odd dimensional foliation on M.

According to the previous lemma, any geodesible 2-dimensional foliation $\mathcal{F}$ on M has to be hyperbolic or elliptic.

Suppose $\mathcal{F}$ is hyperbolic. Then case (1) of Theorem D cannot occur since the Euler characteristic of a sphere bundle over a surface of genus greater than 1 is negative. So, the universal covering space $\tilde{M}$ of M is a product $\mathbf{D}^2 \times \mathbf{R}^2$ and the fundamental group Γ of M acts by isometries on $\mathbf{D}^2$. In other words, we obtain a homomorphism h from Γ to $\mathrm{Isom}(\mathbf{D}^2) \simeq \mathrm{PSL}(2, \mathbf{R})$. The compactness of M implies that $\mathrm{PSL}(2, \mathbf{R})/\overline{h(\Gamma)}$ is compact. But Γ being supposed Abelian, $\overline{h(\Gamma)}$ is an Abelian subgroup of $\mathrm{PSL}(2, \mathbf{R})$ and so is contained in a 1-parameter subgroup of $\mathrm{PSL}(2, \mathbf{R})$. This contradicts the fact that $\mathrm{PSL}(2, \mathbf{R})/\overline{h(\Gamma)}$ is compact. So, $\mathcal{F}$ cannot be hyperbolic and hence has to be elliptic.

By Theorem B, the leaves of $\mathcal{F}$ are the fibers of a fibration by spheres. The base of this fibration must be $\mathbf{S}^2$ because $\chi(M)$ is positive. There are two such fibrations by virtue of the fact that $\pi_1(\mathrm{Diff}^+(\mathbf{S}^2)) = \mathbf{Z}/2\mathbf{Z}$. q.e.d.

The proof of Theorem F is a consequence of Theorems A and B, Lemma 5.1 and the following:

Proposition 5.3. *Let $\mathcal{F}$ be a geodesible hyperbolic foliation on a compact 4-manifold M. Then $\chi(M)$ is nonnegative unless the leaves of $\mathcal{F}$ are transverse to a fibration of M by spheres $\mathbf{S}^2$.*

Proof. By Theorem D, the distribution $\mathcal{F}^{\perp}$ is integrable and defines a foliation $\mathcal{N}$. Suppose that $\mathcal{F}$ is not transverse to a fibration by spheres (case (1) of Theorem D). Then, the universal covering space $\tilde{M}$ of M is a product

$\mathbf{D}^2 \times \mathbf{R}^2$ and the fundamental group Γ of M acts by isometries on $\mathbf{D}^2$. In particular, no leaf of $\mathscr{N}$ is diffeomorphic to $\mathbf{S}^2$.

Let ω be the volume 2-form on the Poincaré disc $\mathbf{D}^2$. By pull-back, we obtain a Γ-invariant 2-form on $\tilde{M}$ and hence a closed 2-form on M, still denoted by ω. This form can be interpreted in two ways. On the one hand, ω naturally defines a transverse invariant measure for $\mathscr{N}$. The cohomology class associated to this transverse measure (cf. [15]) is obviously the cohomology class $[\omega]$ of ω in $H^2(M, \mathbf{R})$. On the other hand, since the curvature of $\mathbf{D}^2$ is -1, the Chern-Weil theorem implies that the Euler class of the tangent bundle of $\mathscr{F}$ is $-[\omega]$. Let e be the Euler class of the tangent bundle of $\mathscr{N}$. The cup product $e \cup (-[\omega])$ is the Euler class of the Whitney sum of the tangent bundles of $\mathscr{N}$ and $\mathscr{F}$, that is the Euler class of the tangent bundle of M. Consequently, the evaluation of $e \cup [\omega]$ on the fundamental class $[M]$ of M is the opposite of the Euler characteristic $\chi(M)$ of M. In order to prove the proposition, that is, to show that $\chi(M)$ is nonnegative, it suffices now to apply the following theorem of [9]. Let $\mathscr{N}$ be an oriented 2-dimensional foliation on an oriented compact manifold M. Suppose that no leaf of $\mathscr{N}$ is diffeomorphic to $\mathbf{S}^2$ and that $\mathscr{N}$ admits a transverse invariant measure ω. Let $[\omega]$ be the cohomology class associated to ω and let e be the Euler class of the tangent bundle of $\mathscr{N}$. Then the evaluation of the cup product $e \cup [\omega]$ on the fundamental class $[M]$ of M is nonpositive. This concludes the proof of Proposition 5.3 and, hence, of Theorem F. q.e.d.

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